Math 132: Differential Topology

§ Manifolds with boundary

Consider the closed half-space

$$H^m := \{(x_1, \cdots, x_m) \in \mathbb{R}^m \mid x_m \geq 0\}.$$

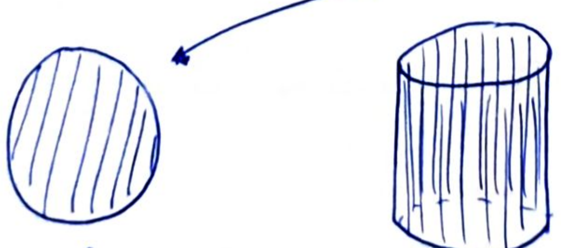
The boundary ∂H^m is defined to be the hyperplane $\mathbb{R}^{m-1} \times 0 \subset \mathbb{R}^m$.

Def A subset $M \subset \mathbb{R}^k$ is called a smooth m-manifold with boundary if it is locally diffeomorphic to H^m.

The boundary ∂M is the set of all points of M which correspond to points of ∂H^m under such a local ~~diffeomorphism~~ parametrization.

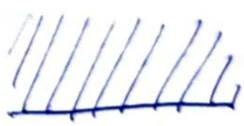
Its complement is called the interior $\mathrm{Int}(M) = M - \partial M$.

e.g.



D^2 $\qquad$ $S^1 \times I$

both are locally diffeomorphic to H^2



Prop If M is a smooth m-manifold with boundary, ∂M is a well-defined smooth $(m-1)$-manifold (<u>without</u> boundary).

proof) For any open neighborhood U of $x \in \partial H^m$ in H^m, $U \cap \partial H^m$ is an open subset of $\partial H^m \cong \mathbb{R}^{m-1}$, so it suffices to show that any diffeomorphism between open subsets of H^m sends boundary points to boundary points.

Suppose $f: U \to V$ is such a diffeomorphism, and that it maps some $x \in \partial U$ to an interior point $y = f(x)$ of V.
But then f^{-1} must map an open an open neighborhood of y in $\mathbb{R}^m$ contained in V onto an open neighborhood of x in $\mathbb{R}^m$, which contradicts $x \in \partial U$.
Therefore, ∂M is a well-defined smooth $(m-1)$-manifold. ■

The <u>tangent space</u> T_pM is defined just as before, so that it is a full m-dimensional vector space, even if $p \in \partial M$.
In case $p \in \partial M$, there is a natural inclusion $\underset{(m-1)\text{-dim'l}}{T_p(\partial M)} \hookrightarrow \underset{m\text{-dim'l}}{T_pM}$.

3/ Here's one method for generating examples:

Lem Let M be a manifold (without boundary) and let $g: M \to \mathbb{R}$ have 0 as a regular value.

Then $\{x \in M \mid g(x) \geq 0\}$ is a smooth manifold with boundary $g^{-1}(0)$.

proof) The set where $g > 0$ is open in M and hence a submanifold of the same dimension. At points of $g^{-1}(0)$, g is locally equivalent to the canonical submersion, and the lemma is obvious. ■

Ex The unit disk $D^m = \{x \in \mathbb{R}^m \mid 1 - \sum_{i=1}^{m} x_i^2 \geq 0\}$ is a smooth manifold with boundary $\partial D^m = S^{m-1}$.

We also have a version of preimage theorem:

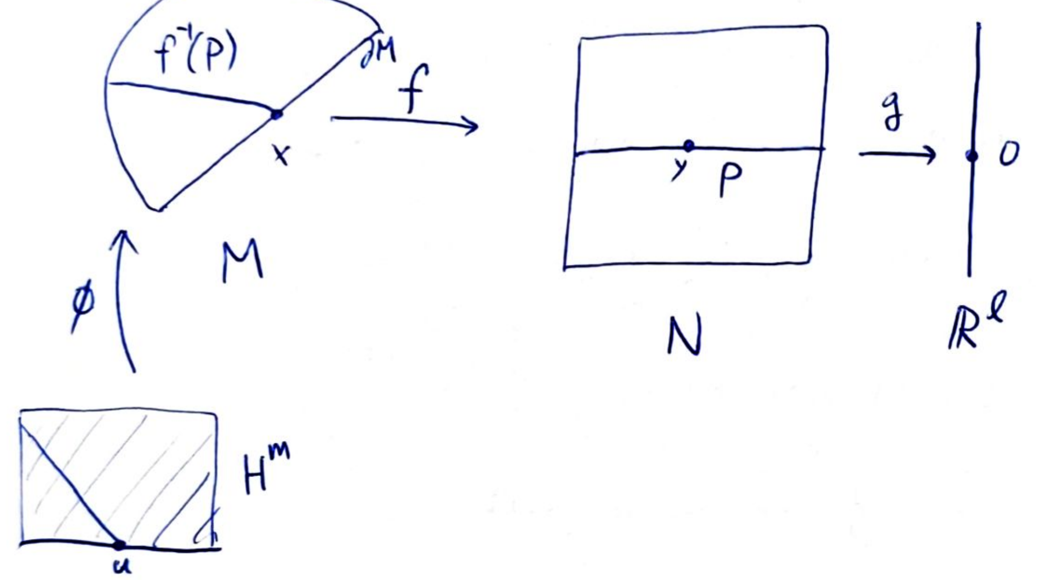
Thm Let $f: M \to N$ be a smooth map from a manifold M with bdry to a manifold N (without boundary), and suppose that both $f: M \to N$ and $\partial f := f|_{\partial M}: \partial M \to N$ are transversal with respect to a submanifold $P \subset N$.

Then, $f^{-1}(P) \subset M$ is a manifold with boundary $\partial f^{-1}(P) = f^{-1}(P) \cap \partial M$ and

$$\operatorname{codim}(f^{-1}(P) \subset M) = \operatorname{codim}(P \subset N).$$

proof) We only need to examine $f^{-1}(P)$ in a neighborhood of a point $x \in f^{-1}(P) \cap \partial M$, thanks to the earlier preimage theorem.

As usual, in some neighborhood of $y = f(x) \in P$, there's a submersion g to $\mathbb{R}^l$, $l = \operatorname{codim}(P \subset N)$, such that $P = g^{-1}(0)$ in this neighborhood.



Pre-composing with a local parametrization $\phi : U \subset H^m \to M$ around x,
$u \mapsto x$

we obtain a map $h = g \circ f \circ \phi : U \subset H^m \to \mathbb{R}^l$ which is a submersion.
$u \mapsto 0$

By definition of smoothness, h extends to a smooth map $\tilde{h}$ defined on a neighborhood $\tilde{U}$ of u in $\mathbb{R}^m$, and the preimage $\tilde{h}^{-1}(0) \subset \tilde{U}$ is a manifold.

Now, the transversality assumption on ∂f implies that the last coordinate x_m is regular on $\tilde{h}^{-1}(0)$ at u. Therefore, by the previous lemma, $\tilde{h}^{-1}(0) \cap \{x_m \geq 0\}$ is a manifold with boundary $\tilde{h}^{-1}(0) \cap \{x_m = 0\}$. ∎